J. Fluid Mech. (1990), *vol.* 217, *pp.* 367-390
Printed in Great Britain

Wave interactions in a three-dimensional attachment-line boundary layer

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(Received 27 August 1988 and in revised form 18 January 1990)

The three-dimensional boundary layer on a swept wing can support different types of hydrodynamic instability. Here attention is focused on the so-called 'spanwise instability' problem which occurs when the attachment-line boundary layer on the leading edge becomes unstable to Tollmien–Schlichting waves. In order to gain insight into the interactions that are important in that problem a simplified basic state is considered. This simplified flow corresponds to the swept attachment-line boundary layer on an infinite flat plate. The basic flow here is an exact solution of the Navier–Stokes equations and its stability to two-dimensional waves propagating along the attachment line can be considered exactly at finite Reynolds number. This has been done in the linear and weakly nonlinear regimes by Hall, Malik & Poll (1984) and Hall & Malik (1986). Here the corresponding problem is studied for oblique waves and their interaction with two-dimensional waves is investigated. In fact oblique modes cannot be described exactly at finite Reynolds number so it is necessary to make a high-Reynolds-number approximation and use triple-deck theory. It is shown that there are two types of oblique wave which, if excited, cause the destabilization of the two-dimensional mode and the breakdown of the disturbed flow at a finite distance from the leading edge. First a low-frequency mode closely related to the viscous stationary crossflow mode discussed by Hall (1986) and MacKerrell (1987) is a possible cause of breakdown. Secondly a class of oblique wave with frequency comparable with that of the two-dimensional mode is another cause of breakdown. It is shown that the relative importance of the modes depends on the distance from the attachment line.

1. Introduction

Our interest is in the interaction of instability waves near the leading edge of a swept wing. The waves we consider are Tollmien–Schlichting modes induced by the viscosity of the basic state. Our investigation of this interaction problem is stimulated by the renewed interest in recent years in the development of laminar-flow wings. The mechanism that we consider occurs near the front of a wing so that it is a probable cause of the onset of transition in the flow. Clearly if this type of disturbance cannot be controlled then there is no point in being concerned about the different instability mechanisms which become operational as the flow develops. A primary aim of this paper is to investigate the possible role of three-dimensional waves propagating at a finite angle to the flow attachment line. In particular we consider the interaction of such oblique modes with the two-dimensional mode which propagates along the attachment line. Before giving a more detailed account of the attachment-line instability problem we shall briefly discuss the other possible instability mechanisms in three-dimensional boundary-layer flows.

The most commonly studied instability mechanism in three-dimensional boundary layers is the so-called 'crossflow vortex' instability first investigated theoretically in any detail by Gregory, Stuart & Walker (1955). They considered the boundary layer on a rotating disc and formulated the instability equations for inviscid disturbances of wavelength comparable with the boundary-layer thickness. The fact that the wavelength scales like the boundary-layer thickness means that non-parallel effects can be ignored so that the instability problem reduces to Rayleigh's equation, which is a second-order differential equation. In general this equation has a singular point if the wave speed of the disturbance turns out to be real. In fact the crossflow vortex mode is stationary and the effective basic flow has an inflection point at the critical layer so that there is no singularity there. Apart from this inviscid stationary mode there is a viscous stationary mode governed by triple-deck theory, see Hall (1986), MacKerrell (1987). However, the growth rates of the inviscid disturbances are bigger than those of the viscous ones so that inviscid modes are probably the most important in any practical situation. But since this mechanism is not operational near the attachment line, the mode considered in this paper is the most likely cause of transition. It should also be noted that unsteady inviscid crossflow vortices are possible and can in fact have the largest growth rates; presumably in any experiment all types of modes could be excited but the flow visualization techniques used in the early experimental investigations of the problem pointed to the particular importance of stationary modes so they have received the most theoretical attention. The interaction of stationary crossflow-vortices with viscous Tollmien–Schlichting waves has recently been considered by Bassom & Hall (1990).

In the presence of surface curvature the Görtler vortex mechanism can also be a possible cause for the onset of transition. This mechanism is caused by centrifugal effects and might be particularly important in the flow over the concave region of modern laminar-flow airfoils which have significant regions of concave curvature. However, it was shown by Hall (1986) that in a three-dimensional boundary layer this mechanism is destroyed by an asymptotically small crossflow velocity field so it could well be that in the practical situation it is unimportant.

Now let us return to a discussion of previous work on the attachment-line instability problem. First we note that near the leading edge of an infinite swept cylinder the boundary layer can be approximated in an asymptotic sense by a stagnation-point boundary layer in the plane normal to the attachment line together with a parallel boundary layer along the attachment line. This approximation is of course only valid close to the leading edge and corresponds to the three-dimensional boundary layer on an infinite plate inclined at an angle to an oncoming flow. It is the latter flow which is usually used as an approximation to the boundary layer near the leading edge of a wing; we therefore make a similar assumption and confine our attention to the flow past an infinite swept flat plate. The flow therefore consists of a stagnation-point boundary layer in which the chord-wise velocity component increases with distance from the attachment line along which a 'parasitic' constant-thickness boundary layer exists.

The notable experimental investigations of this problem are due to Pfenninger & Bacon (1969), Gaster (1967) and Poll (1979). These investigators were concerned with different cylinder shapes and confined their attention to positions very close to the attachment line. It was found that small-amplitude Tollmien–Schlichting waves propagate along the attachment line if the input disturbance is small. However, if the input disturbance is large enough the results of Pfenninger & Bacon (1969) suggest that there might be a subcritical response by the flow. In all of these investigations measurements were made close to the attachment line before the three-dimensional breakdown of the flow considered here could take place. Though this three-dimensional breakdown has not been investigated experimentally we feel that it is possibly of great importance and might relate to the three-dimensional crossflow modes which have been measured further downstream in the chordwise direction.

The experimental work discussed above was confirmed by the linear stability calculation of Hall, Malik & Poll (1984) and the weakly nonlinear and numerical simulations of Hall & Malik (1986); hereinafter we refer to these papers as HMP and HM respectively. In the linear case HMP solved the linear stability problem for modes propagating along the attachment line. These modes are described by ordinary differential equations at all values of the Reynolds numbers so that HMP were able to find the whole of the neutral curve which separates the stable and unstable modes. Almost all of the experimental points were found to be on the lower branch of the neutral curve; subsequently HM showed that these modes bifurcate supercritically whilst the upper branch modes bifurcate subcritically. Thus the linear, weakly nonlinear and experimental work were found to be in excellent agreement; the subcritical nature of the upper branch found by HM is also consistent with Pfenninger & Bacon's experimental observations of finite-amplitude disturbances below the critical Reynolds number. Some evidence of this kind of response was found by HM in their full numerical simulations; the supercritical nature of the lower branch was also confirmed by the numerical simulations of HM.

There is of course no reason why Tollmien–Schlichting waves propagating at an angle to the attachment line cannot be important in the region where the basic state develops in the chordwise direction. It appears that no previous work on this problem has been carried out so here we discuss the linear and weakly nonlinear regimes for such disturbances in the presence of a two-dimensional mode. In fact the three-dimensional modes are significantly different from the two-dimensional ones because they have a chordwise dependence different from that of the basic state. This means that, if we wish to construct a stability theory connected in some asymptotic sense to that appropriate to the full Navier–Stokes equations, we must restrict our attention to the high-Reynolds-number limit. In view of this fact, and because experimental work close to the attachment line suggests that lower-branch modes are the ones observed experimentally, we use triple-deck theory to investigate the stability properties of three-dimensional modes. This is done in the linear regime following the work of Smith (1979*a*) whilst in the nonlinear regime, where the interaction of modes is considered, we use the formulation of Hall & Smith (1984); this formulation also allows non-parallel effects to be taken care of in a self-consistent asymptotic manner. We shall show that three-dimensional interactions are possible which trigger a significant large-amplitude response which cannot be stimulated by the two-dimensional mode. We identify the orientation of the waves which produces this and other significant responses. The procedure adopted in the rest of this paper is as follows: in §2 we formulate the instability problem and discuss the solution of the linearized stability equations. In §3 we derive the amplitude equations which describe the interaction of a two-dimensional wave with an oblique wave. The solution of these amplitude equations is discussed in §4 whilst in §5 we draw some conclusions.

2. Formulation of the stability problem

Consider the flow of a viscous incompressible fluid of kinematic viscosity ν adjacent to the flat plate defined by $y = 0$ with respect to Cartesian coordinates (x, y, z). The velocity field (u, v, w) corresponding to (x, y, z) satisfies the conditions

$$u = v = w = 0, \quad y = 0, \tag{2.1a}$$

$$u \to U_\infty \frac{x}{l}, \quad w \to U_\infty, \quad y \to \infty, \tag{2.1b}$$

where U_∞ is a typical free-stream velocity and l is a typical lengthscale in the x-direction. The Reynolds number R is defined by

$$R = \frac{U_\infty l}{\nu}, \tag{2.2}$$

and dimensionless variables $(X, Y, Z), (U, V, W)$ are defined by

$$(X, Y, Z) = \left(\frac{x}{l}, \frac{y}{l}, \frac{z}{l}\right), \tag{2.3a}$$

$$(u, v, w) = U_\infty(U, V, W). \tag{2.3b}$$

If the pressure is scaled on ρU_∞^2 and time T on U_∞/l, where ρ is the density of the fluid, then the Navier–Stokes equations and the continuity equation become

$$\boldsymbol{U}_T + (\boldsymbol{U}\cdot\boldsymbol{\nabla})\,\boldsymbol{U} = -\boldsymbol{\nabla} p + \frac{1}{R}\nabla^2 \boldsymbol{U}, \tag{2.4a}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{U} = 0, \tag{2.4b}$$

where P is the non-dimensional pressure. In order to satisfy the no-slip condition the basic state has a boundary layer of thickness $lR^{-\frac{1}{2}}$ near $Y = 0$. If the boundary-layer variable η is defined by

$$\eta = R^{\frac{1}{2}} Y,$$

the basic state can be expressed in the form

$$\boldsymbol{U} = (X\bar{u}(\eta), R^{-\frac{1}{2}}\bar{v}(\eta), \bar{w}(\eta))\,(1 + O(R^{-\frac{1}{2}})), \tag{2.5}$$

where

$$\bar{u} + \bar{v}' = 0, \tag{2.6a}$$

$$\bar{v}''' + \bar{v}'^2 - \bar{v}\bar{v}'' - 1 = 0, \tag{2.6b}$$

$$\bar{w}'' - \bar{v}\bar{w}' = 0, \tag{2.6c}$$

with

$$\bar{v}(0) = \bar{v}'(0) = \bar{w}(0) = 0, \tag{2.7a}$$

$$\bar{v}'(\infty) = -1, \quad \bar{w}(\infty) = 1. \tag{2.7b}$$

It is known, Smith (1979a, b) that lower-branch Tollmien–Schlichting instabilities of boundary layers are governed by triple-deck theory. Here the interest is in oblique Tollmien–Schlichting waves proportional to E where

$$E = \exp\left[\mathrm{i}\left\{\int^X \frac{\alpha(X)}{\epsilon^3}\,\mathrm{d}X + \frac{\beta Z}{\epsilon^3} - \frac{\Omega T}{\epsilon^2}\right\}\right], \tag{2.8}$$

with $\epsilon = R^{-\frac{1}{8}} \ll 1$. The triple-deck structure has upper, main and lower decks of thicknesses $O(\epsilon^3)$, $O(\epsilon^4)$, and $O(\epsilon^5)$, respectively. The normal velocity is scaled appropriately in each deck, while the X and Z scales are of the same order as the thickness of the upper deck. The timescale is chosen so that

$$\frac{\partial \boldsymbol{U}}{\partial t} \sim \boldsymbol{U}\frac{\partial \boldsymbol{U}}{\partial X},$$

in the lower deck. The slowly varying wavenumber α is then expanded as

$$\alpha = \alpha_0 + \epsilon\alpha_1 + \dots. \tag{2.9}$$

In the main part of the boundary layer the basic state is perturbed by writing

$$\boldsymbol{U} = (X\bar{u}, \epsilon^4\bar{v}, \bar{w})(1+O(\epsilon^4)) + \delta(\epsilon U_0, \epsilon^2 V_0, \epsilon W_0)E + \dots, \tag{2.10}$$

where U_0, V_0 and W_0 depend only on X and η whilst δ is assumed small compared to any power of ϵ. The corresponding pressure perturbation is $\epsilon^2 P_0 E$, where P_0 is a function of X only. The equations to determine U_0, V_0 in the main deck are:

$$\mathrm{i}\alpha_0 U_0 + V_{0\eta} + \mathrm{i}\beta W_0 = 0, \tag{2.11a}$$

$$\mathrm{i}\alpha_0 X\bar{u}U_0 + XV_0\bar{u}' + \mathrm{i}\beta\bar{w}U_0 = 0, \tag{2.11b}$$

$$\mathrm{i}\alpha_0 X\bar{u}W_0 + V_0\bar{w}' + \mathrm{i}\beta\bar{w}W_0 = 0, \tag{2.11c}$$

and the appropriate solution of this system is

$$U_0 = a(X)X\bar{u}', \quad W_0 = a(X)\bar{w}', \tag{2.12a, b}$$

$$V = -\mathrm{i}a(X)[\alpha_0 X\bar{u} + \beta\bar{w}], \quad P_0 = a(X), \tag{2.12c, d}$$

where $a(X)$ is an amplitude function to be determined. Note that the solution we have found could be obtained by transforming to Squire coordinates. An investigation of the disturbed flow in the upper layer shows that all disturbance quantities decay exponentially there. In particular the pressure perturbation is $\epsilon^2\hat{P}_1 E$ where

$$\hat{P}_1 = \frac{a(\alpha_0 X + \beta)^2}{(\alpha_0^2+\beta^2)^{\frac{1}{2}}}\exp\left[-(\alpha_0^2+\beta^2)^{\frac{1}{2}}\hat{Y}\right], \tag{2.13a}$$

where $Y = \epsilon^3\hat{Y}$. Thus matching with the main-deck solution requires

$$P_0(\alpha_0^2+\beta^2)^{\frac{1}{2}} = a[\alpha_0 X + \beta]^2. \tag{2.13b}$$

It should be noted from (2.12) that when $\eta \to 0$

$$U_0 \sim a(X)\lambda X, \quad W_0 \sim a(X)\mu, \tag{2.14}$$

where from numerical calculations it is found that $\lambda = 1.236$, $\mu = 0.570$. Finally, in the lower deck the pressure perturbation is still $\epsilon^2 P_0 E$, i.e. the pressure perturbation is independent of ζ, whilst the total velocity field can be written:

$$U = \epsilon\zeta\lambda X + \dots + \delta\epsilon u_0 E + \dots, \tag{2.15a}$$

$$V = -\tfrac{1}{2}\epsilon^6\zeta^2\lambda + \dots + \delta\epsilon^3 v_0 E + \dots, \tag{2.15b}$$

$$W = \epsilon\zeta\mu + \dots + \delta\epsilon w_0 E + \dots. \tag{2.15c}$$

Here the lower-deck variable ζ is defined by

$$\zeta = Y\epsilon^{-5}.$$

The equations to determine (u_0, v_0, w_0) in the lower deck can be written as

$$\mathbf{L}\begin{pmatrix} u_0 \\ v_0 \\ w_0 \\ P_0 \end{pmatrix} = 0, \tag{2.16}$$

where the matrix operator

$$\mathbf{L}(\alpha_0, \beta, \Omega) \equiv \begin{pmatrix} \dfrac{\mathrm{d}^2}{\mathrm{d}\zeta^2} - \mathrm{i}\alpha_0 X\lambda\zeta - \mathrm{i}\beta\mu\zeta + \mathrm{i}\Omega & -\lambda X & 0 & -\mathrm{i}\alpha_0 \\ \mathrm{i}\alpha_0 & \dfrac{\mathrm{d}}{\mathrm{d}\zeta} & \mathrm{i}\beta & 0 \\ 0 & -\mu & \dfrac{\mathrm{d}^2}{\mathrm{d}\zeta^2} - \mathrm{i}\alpha_0 X\lambda\zeta - \mathrm{i}\beta\mu\zeta + \mathrm{i}\Omega & -\mathrm{i}\beta \end{pmatrix} \tag{2.17}$$

The boundary conditions are that u_0, v_0 and w_0 satisfy the no-slip condition at the wall and match with the main-deck solution as $\zeta \to \infty$. The solution for $\alpha_0 u_0 + \beta w_0$ and v_0 is obtained from (2.16) in the following manner. We multiply the first equation of (2.16) by α_0 and the second one by β and add the resulting equations together. The resulting equation is differentiated with respect to ζ to eliminate P_0 and v_0 is substituted from the second equation of (2.16). The final equation is an Airy equation for $\alpha_0 \partial u_0/\partial\zeta + \beta\, \partial w_0/\partial\zeta$. Thus, the solution of the wall-layer is then written as

$$\alpha_0 u_0 + \beta w_0 = b(X) \int_{\xi_0}^{\xi} \mathrm{Ai}\,(s)\,\mathrm{d}s, \tag{2.18}$$

$$\varDelta v_0 = -\mathrm{i} \int_{\xi_0}^{\xi} (\alpha_0 u_0 + \beta w_0)\,\mathrm{d}s, \tag{2.19}$$

where Ai is the Airy function and

$$\varDelta = \{\mathrm{i}\lambda X\alpha_0 + \mathrm{i}\beta\mu\}^{\frac{1}{3}}, \tag{2.20a}$$

$$\xi = \varDelta\zeta + \xi_0, \tag{2.20b}$$

$$\xi_0 = -\frac{\mathrm{i}\Omega}{\varDelta^2}. \tag{2.20c}$$

The pressure P_0 can be related to the displacement function b by using the X and Z momentum equations to give

$$\mathrm{i}\{\alpha_0^2 + \beta^2\} P_0 = \varDelta^2 \,\mathrm{Ai}'\,(\xi_0)\, b. \tag{2.21}$$

The lower- and main-deck solutions are then found to match if

$$\mathrm{Ai}'\,(\xi_0)\,(\alpha_0 \lambda X + \beta\mu)^{\frac{5}{3}} = \mathrm{i}^{\frac{1}{3}}(\alpha_0^2 + \beta^2)^{\frac{1}{2}} \chi_0 (\alpha_0 X + \beta)^2, \tag{2.22}$$

where
$$\chi_0 = \int_{\xi_0}^{\infty} \mathrm{Ai}\,(s)\,\mathrm{d}s.$$

The eigenrelation (2.22) determines the complex wavenumber α_0 for given β and Ω. For neutral stability it is known that

$$\xi_0 \sim -2.298\mathrm{i}^{\frac{1}{3}}, \quad \frac{\mathrm{Ai}'\,(\xi_0)}{\chi_0} \sim 1.001\mathrm{i}^{\frac{1}{3}}. \tag{2.23a, b}$$

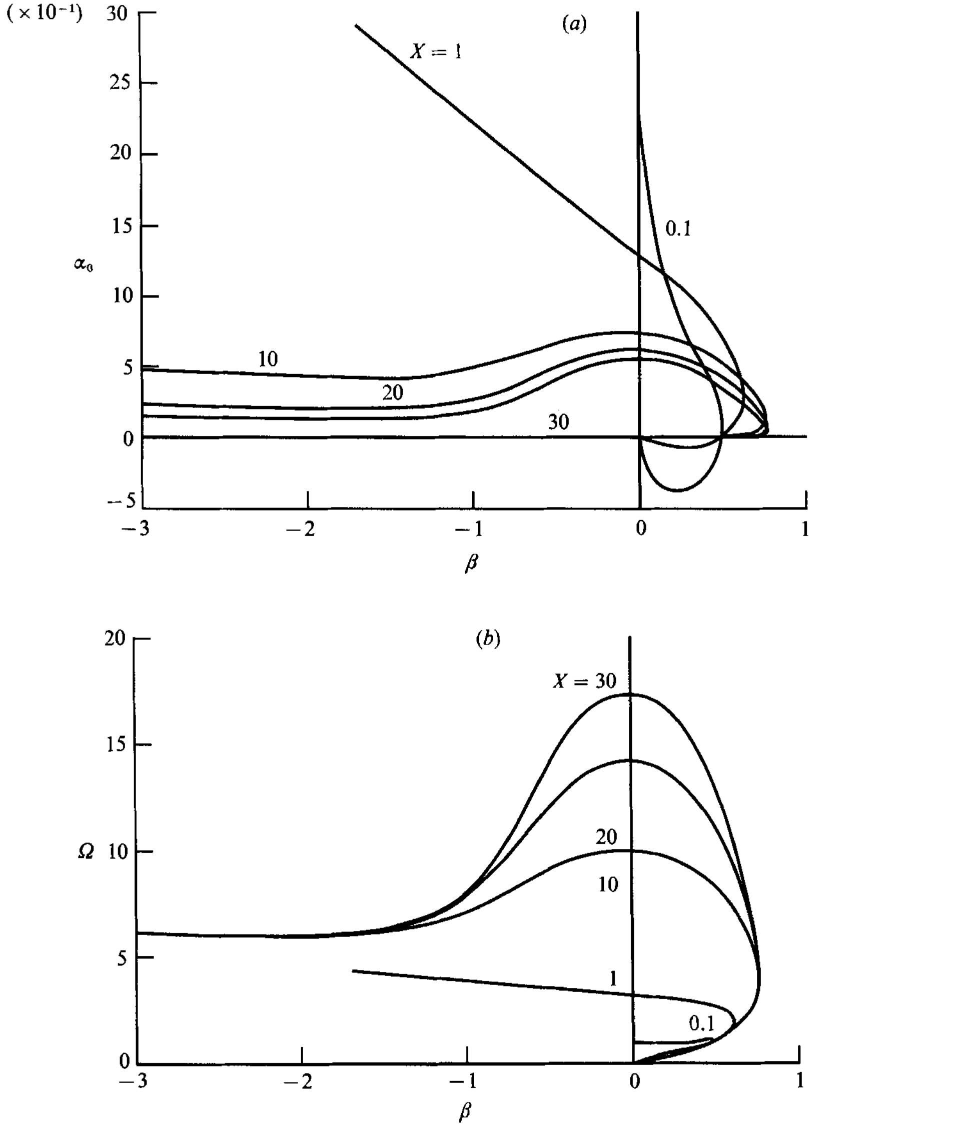

FIGURE 1. The neutral eigenvalues satisfying (2.24).

Thus the neutral values of α_0 and β are related by

$$1.001(\alpha_0 \lambda X+\beta\mu)^{\frac{5}{3}} \sim (\alpha_0^2+\beta^2)^{\frac{1}{2}}(\alpha_0 X+\beta)^2. \tag{2.24}$$

The modes corresponding to those of HMP have $\alpha_0 = 0$ in which case β satisfies

$$\beta^{\frac{4}{3}} \sim 1.001\mu^{\frac{5}{3}}.$$

At any given value of X there are in addition neutral three-dimensional modes with $\alpha_0 \neq 0$. The above analysis fails if $\alpha_0 X+\beta$ or $\alpha_0 \lambda X+\beta\mu$ become negative anywhere in the (α_0, β)-plane so only eigenvalues of (2.24) above the lines $\alpha_0 \lambda X+\beta\mu = 0$, and $\alpha_0 X+\beta = 0$ are acceptable. In figure 1(*a*) α_0 is shown as a function of β for $X = 0.1$, 1, 10, 20, 30. The solutions in the second quadrant asymptote to the line

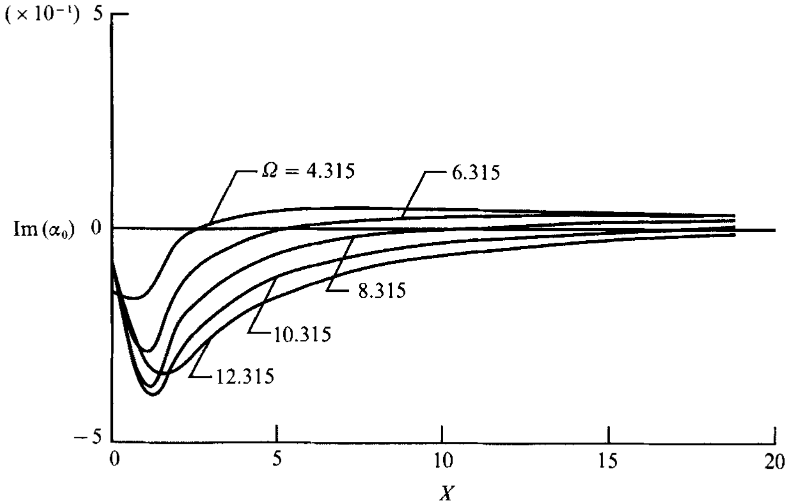

FIGURE 2. The imaginary part of α_0 satisfying (2.24), as a function of X for a range of values of Ω.

$\alpha_0 X+\beta = 0$ as $\beta \rightarrow -\infty$ whilst those in the fourth quadrant asymptote to the line $\alpha_0 \lambda X+\beta\mu = 0$ as $\beta \rightarrow 0$. Figure 1(*b*) shows the neutral values of Ω from (2.20) as a function of β for $X = 0.1$, 1, 10, 20, 30.

Finally in this section we notice that the two-dimensional mode of (2.22), which of course corresponds to $\alpha_0 = 0$, is neutrally stable at all values of X. The three-dimensional modes, however, are initially unstable on the attachment line $X = 0$ and become stable beyond a critical value of X. Experimentally it appears that if the level of disturbances present in the flow is sufficiently small then it is the two-dimensional mode which is observed. In the next section we investigate the possibility that the two-dimensional mode might be destabilized by oblique modes which grow in the X-direction. In figure 2 we have shown typical growth rate curves for the three-dimensional modes for $\beta = 0.5$.

In fact the small-β solutions are related to the stationary modes of instability of the three-dimensional boundary layer discussed by Hall (1986) and MacKerrell (1987). These modes orient themselves such that the shear stress of the 'effective' velocity profile is zero; the lower-deck structure is then described by parabolic cylinder functions rather than Airy functions. Thus when α_0 tends to zero we find from (2.24) that

$$\mu\beta = -\alpha_0 \lambda X + O(\alpha_0)^{\frac{9}{5}},$$

so that the neutral frequency tends to zero like $\alpha_0^{\frac{6}{5}}$. The time-dependent version of the stationary modes discussed by Hall and MacKerrell has recently been considered by Bassom & Gajjar (1988).

3. Weakly nonlinear theory

Suppose that the three-dimensional mode with $(\alpha, \beta, \Omega) = (\alpha_2, \beta_2, \Omega_2)$ is neutrally stable at $X = X_n$. We consider the interaction of this mode with the two-dimensional disturbance which propagates along the attachment line. We know from the work of Hall & Smith (1984) that in the absence of the two-dimensional mode the three-dimensional mode will evolve in a nonlinear, non-parallel manner in an $\epsilon^{\frac{3}{2}}$ neighbourhood of X_{n}. We therefore define $\tilde{X}$ by

$$\tilde{X} = \frac{(X - X_{\mathrm{n}})}{\epsilon^{\frac{3}{2}}}. \tag{3.1}$$

Later we can derive the 'quasi-parallel' evolution equations for $(X - X_{\mathrm{n}}) > O(\epsilon^{\frac{3}{2}})$ by taking the limit $\tilde{X} \to \infty$. In order that the two-dimensional mode in this neighbourhood should be of finite size we suppose that, with $\Omega = \Omega_1$, the neutral frequency for a two-dimensional wave, the spanwise wavenumber β_1 is expanded as

$$\beta_1 = \beta_{10} + \epsilon\beta_{11} + \epsilon^{\frac{3}{2}}\tilde{\beta} + \dots, \tag{3.2}$$

where β_{10}, β_{11} are the first two terms in the expansion of the neutral spanwise wavenumber.

It is now convenient to represent the 'fast' dependence of the Tollmien–Schlichting waves in the X-direction by multiple scales rather than the WKB formulation of §2. We therefore write

$$X^* = \frac{(X - X_{\mathrm{n}})}{\epsilon^3}. \tag{3.3}$$

Next we define

$$E_1 = \exp\left[\mathrm{i}\left\{\frac{\beta_1 Z}{\epsilon^3} - \frac{\Omega_1 T}{\epsilon^2}\right\}\right],$$

$$E_2 = \exp\left[\mathrm{i}\left\{\alpha_2 X^* + \frac{\beta_2 Z}{\epsilon^3} - \frac{\Omega_2 T}{\epsilon^2}\right\}\right],$$

where Ω_1, Ω_2 and β_2 expand as

$$\Omega_1 = \Omega_{10} + \epsilon\Omega_{11} + O(\epsilon^2), \tag{3.4a}$$

$$\Omega_2 = \Omega_{20} + \epsilon\Omega_{21} + O(\epsilon^2), \tag{3.4b}$$

$$\beta_2 = \beta_{20} + \epsilon\beta_{21} + O(\epsilon^2). \tag{3.4c}$$

Here Ω_{10}, Ω_{11} etc. are the neutral values appropriate to the location $X = X_{\mathrm{n}}$ whilst β_1 is as given by (3.2). For the three-dimensional mode we further expand

$$\alpha_2 = \alpha_{20} + \epsilon\alpha_{21} + O(\epsilon^2), \tag{3.4d}$$

where α_{20}, α_{21} are the first two terms in the expansion of the neutral value of α_2 at $X = X_{\mathrm{n}}$. In the lower deck we write the velocity in the form

$$U = \epsilon\zeta\lambda X + \dots + \epsilon\hat{U}, \tag{3.5a}$$

$$V = -\tfrac{1}{2}\epsilon^6\zeta^2\lambda + \dots + \epsilon^3\hat{V}, \tag{3.5b}$$

$$W = \epsilon\zeta\mu + \dots + \epsilon\hat{W}, \tag{3.5c}$$

and then expand the disturbance velocity field $(\hat{U}, \hat{V}, \hat{W})$ together with the corresponding pressure perturbation as

$$(\hat{U}, \hat{V}, \hat{W}, \hat{P}) = \epsilon^{\frac{3}{4}}\boldsymbol{S}_1 + \epsilon^{\frac{3}{2}}\boldsymbol{S}_2 + \epsilon^{\frac{7}{4}}\boldsymbol{S}_3 + \epsilon^{\frac{9}{4}}\boldsymbol{S}_4 + \dots. \tag{3.6}$$

Here the term $\boldsymbol{S}_1$ corresponds to the fundamental modes proportional to E_1 and E_2. The second-order term $\boldsymbol{S}_2$ corresponds to first harmonic and mean flow correction terms generated by the interaction of the Tollmien–Schlichting waves. The third-order term $\boldsymbol{S}_3$ again contains the fundamentals generated because the correction terms in (3.4) are $O(\epsilon)$. Finally the fourth-order term $\boldsymbol{S}_4$ contains fundamental and other terms driven by the interaction of $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$. Note that the scalings in (3.6) have been chosen so that non-parallel and nonlinear effects are equally important.

Clearly the function $\boldsymbol{S}_1$ satisfies the linearized problem of §2 so we write

$$\boldsymbol{S}_1 = A\boldsymbol{S}_{11}E_1 + B\boldsymbol{S}_{12}E_2 + \text{C.C.} \tag{3.7}$$

where C.C. denotes complex conjugate and A and B are functions of $\tilde{X}$ to be found at higher order. The functions $\boldsymbol{S}_{ij}$ are defined by $\boldsymbol{S}_{ij} = (U_{ij}, V_{ij}, W_{ij}, P_{ij})$ for $i,j \geqslant 1$, and $\boldsymbol{S}_{11}, \boldsymbol{S}_{12}$ satisfy (2.16) with $(\alpha_0, \beta, \Omega)$ replaced by $(0, \beta_{10}, \Omega_{10})$ and $(\alpha_{20}, \beta_{20}, \Omega_{20})$ respectively. Thus for example we can show that

$$\frac{\partial P_{12}}{\partial \zeta} = 0,$$

$$\alpha_{20}U_{12} + \beta_{20}W_{12} = c\int_{\xi_{02}}^{\xi_2} \mathrm{Ai}\,(s)\,\mathrm{d}s,$$

$$\xi_{20} = -\frac{\mathrm{i}\Omega_{20}}{\Delta_2^2}, \quad \xi_2 = \Delta_2\zeta + \xi_{20}, \quad \Delta_2 = \{\mathrm{i}\lambda X_n\alpha_{20} + \mathrm{i}\beta_{20}\mu\}^{\frac{1}{3}},$$

where c is a function of $\tilde{X}$.

At next order we find that the first harmonics and mean flow corrections can be written as

$$\boldsymbol{S}_2 = \{A^2\boldsymbol{S}_{21}E_1^2 + B^2\boldsymbol{S}_{22}E_2^2 + AB\boldsymbol{S}_{23}E_1E_2 + A\bar{B}\boldsymbol{S}_{24}E_1E_2^{-1}\} + \text{C.C.} + |A|^2\boldsymbol{S}_{25} + |B|^2\boldsymbol{S}_{26}, \tag{3.8}$$

where $\bar{B}$ denotes the complex conjugate of B. We find that $\boldsymbol{S}_{21}, \boldsymbol{S}_{22}$ satisfy the differential system

$$\mathbf{L}(2\alpha_{n0}, 2\beta_{n0}, 2\Omega_{n0})\begin{pmatrix} U_{2n} \\ V_{2n} \\ W_{2n} \\ P_{2n} \end{pmatrix} = \begin{pmatrix} \mathrm{i}\alpha_{n0}U_{1n}^2 + V_{1n}\dfrac{\mathrm{d}U_{1n}}{\mathrm{d}\zeta} + \mathrm{i}\beta_{n0}W_{1n}U_{1n} \\ 0 \\ \mathrm{i}\alpha_{n0}U_{1n}W_{1n} + V_{1n}\dfrac{\mathrm{d}W_{1n}}{\mathrm{d}\zeta} + \mathrm{i}\beta_{n0}W_{1n}^2 \end{pmatrix}, \tag{3.9}$$

and
$$\frac{\partial P_{2n}}{\partial \zeta} = 0,$$

for $n = 1, 2$ and $\alpha_{10} = 0$ where the operator $\mathbf{L}$ is defined by (2.17). These equations must be solved subject to

$$U_{2n} = V_{2n} = W_{2n} = 0, \quad \zeta = 0, \tag{3.10}$$

and the solutions must match with the main-deck solutions as $\zeta \to \infty$.

The functions $\boldsymbol{S}_{23}$ and $\boldsymbol{S}_{24}$ satisfy similar equations but with $(2\alpha_{n0}, 2\beta_{n0}, 2\Omega_{n0})$ replaced by $(\alpha_{10} \pm \alpha_{20}, \beta_{10} \pm \beta_{20}, \Omega_{10} \pm \Omega_{20})$ for $n = 3, 4$ and the right-hand side of (3.9) replaced respectively by

$$\begin{pmatrix} \mathrm{i}\alpha_{20}U_{11}U_{12} + V_{12}\dfrac{\mathrm{d}U_{11}}{\mathrm{d}\zeta} + V_{11}\dfrac{\mathrm{d}U_{12}}{\mathrm{d}\zeta} + \mathrm{i}(\beta_{10}W_{12}U_{11} + \beta_{20}W_{11}U_{12}) \\ 0 \\ \mathrm{i}\alpha_{20}U_{11}W_{12} + V_{12}\dfrac{\mathrm{d}W_{11}}{\mathrm{d}\zeta} + V_{11}\dfrac{\mathrm{d}W_{12}}{\mathrm{d}\zeta} + \mathrm{i}(\beta_{10} + \beta_{20})W_{11}W_{12} \end{pmatrix},$$

and
$$\begin{pmatrix} -\mathrm{i}\alpha_{20} U_{11} \bar{U}_{12} + \bar{V}_{12} \dfrac{\mathrm{d}U_{11}}{\mathrm{d}\zeta} + V_{11} \dfrac{\mathrm{d}\bar{U}_{12}}{\mathrm{d}\zeta} + \mathrm{i}(\beta_{10} \bar{W}_{12} U_{11} + \beta_{20} W_{11} \bar{U}_{12}) \\ 0 \\ -\mathrm{i}\alpha_{20} U_{11} \bar{W}_{12} + \bar{V}_{12} \dfrac{\mathrm{d}W_{11}}{\mathrm{d}\zeta} + V_{11} \dfrac{\mathrm{d}\bar{W}_{12}}{\mathrm{d}\zeta} + \mathrm{i}(\beta_{10} - \beta_{20}) W_{11} \bar{W}_{12} \end{pmatrix}.$$

The mean flow corrections for $\boldsymbol{S}_{25}$ and $\boldsymbol{S}_{26}$ have $V_{25} = V_{26} = 0$ whilst for $n = 1, 2$

$$\frac{\mathrm{d}^2 U_{2n+4}}{\mathrm{d}\zeta^2} = V_{1n} \frac{\mathrm{d}\bar{U}_{1n}}{\mathrm{d}\zeta} + \bar{V}_{1n} \frac{\mathrm{d}U_{1n}}{\mathrm{d}\zeta} + \mathrm{i}\beta_{n0}(\bar{W}_{1n} U_{1n} - W_{1n} \bar{U}_{1n}), \tag{3.11a}$$

$$\frac{\mathrm{d}^2 W_{2n+4}}{\mathrm{d}\zeta^2} = \mathrm{i}\alpha_{n0}(\bar{U}_{1n} W_{1n} - U_{1n} \bar{W}_{1n}) + V_{1n} \frac{\mathrm{d}\bar{W}_{1n}}{\mathrm{d}\zeta} + \bar{V}_{1n} \frac{\mathrm{d}W_{1n}}{\mathrm{d}\zeta}, \tag{3.11b}$$

which must be solved subject to

$$U_{2n+4} = W_{2n+4} = 0, \quad \zeta = 0, \tag{3.12a}$$

$$\frac{\mathrm{d}U_{2n+4}}{\mathrm{d}\zeta} = \frac{\mathrm{d}W_{2n+4}}{\mathrm{d}\zeta} = 0, \quad \zeta = \infty. \tag{3.12b}$$

In the main and upper decks the mean flow corrections and the first harmonic functions are not forced by the fundamentals and therefore satisfy similar equations to those discussed in §2; the matching of the main-deck solutions for the first harmonics with the lower-deck solution produces boundary conditions at $\zeta = \infty$ for $\boldsymbol{S}_{21}, \boldsymbol{S}_{22}, \boldsymbol{S}_{23}, \boldsymbol{S}_{24}$.

At next order in the lower-deck problem we obtain only fundamental terms driven by the variation of the mean state. In fact the solution of this linear problem when matched with the main-deck solution determines the $O(\epsilon)$ terms in the expansion of the neutral wavenumber and frequencies. Since the solution at this order has no effect on the amplitude equations for A and B we give no details of it here.

The interaction of the fundamental term $\boldsymbol{S}_1$ with the mean flow correction and first harmonic term generates fundamental terms in $\boldsymbol{S}_4$. In addition further fundamental terms are produced by the evolution of the amplitude functions A and B and the basic state on the $\tilde{X}$ lengthscale. If we write $\boldsymbol{S}_4$ in the form

$$\boldsymbol{S}_4 = \boldsymbol{S}_{41} E_1 + \boldsymbol{S}_{42} E_2 + \text{C.C.} + \dots,$$

where ... represents other terms forced by the interactions, then after some manipulation we find that $\boldsymbol{S}_{41}$ and $\boldsymbol{S}_{42}$ satisfy

$$\mathbf{L}(0, \beta_{10}, \Omega_{10}) \begin{pmatrix} U_{41} \\ V_{41} \\ W_{41} \\ P_{41} \end{pmatrix} = \begin{pmatrix} [A|A|^2 \Phi_1 + A|B|^2 \Phi_2 + \mathrm{i}\mu\tilde{\beta}\zeta U_{11} A \\ + \lambda\tilde{X} V_{11} A + \lambda X_{\mathrm{n}} \zeta U_{11} A_{\tilde{X}} + P_{11} A_{\tilde{X}}] \\ -U_{11} A_{\tilde{X}} - \mathrm{i} W_{11} \tilde{\beta} A \\ [A|A|^2 \Phi_3 + A|B|^2 \Phi_4 + \mathrm{i}\tilde{\beta}\mu\zeta W_{11} A \\ + \lambda\zeta X_{\mathrm{n}} W_{11} A_{\tilde{X}} + \mathrm{i}\beta P_{11} A] \end{pmatrix}, \tag{3.13a}$$

and

$$\mathbf{L}(\alpha_{20}, \beta_{20}, \Omega_{20}) \begin{pmatrix} U_{42} \\ V_{42} \\ W_{42} \\ P_{42} \end{pmatrix} = \begin{pmatrix} [B|B|^2 \Phi_5 + B|A|^2 \Phi_6 + \lambda X_{\mathrm{n}} \zeta B_{\tilde{X}} U_{12} \\ + \mathrm{i}\alpha_{20} \tilde{X}\zeta B U_{12} + \lambda\tilde{X} B V_{12} + P_{12} B_{\tilde{X}}] \\ -U_{12} B_{\tilde{X}} \\ [B|B|^2 \Phi_7 + B|A|^2 \Phi_8 + \lambda\zeta X_{\mathrm{n}} W_{12} B_{\tilde{X}} \\ + \lambda\zeta\tilde{X} W_{12} \mathrm{i}\alpha_{20} B] \end{pmatrix}, \tag{3.13b}$$

with $\partial P_{41}/\partial\zeta = \partial P_{42}/\partial\zeta = 0$. Here the functions Φ_1, Φ_2, Φ_3 and Φ_4 are defined by

$$\begin{aligned}\Phi_1 &= \mathrm{i}\alpha_{10}(U_{25}U_{11}+U_{21}\bar{U}_{11})+V_{11}U'_{25}+V_{21}\bar{U}'_{11}+\bar{V}_{11}U'_{21}\\ &\quad+\mathrm{i}\beta_{10}(W_{25}U_{11}-W_{21}\bar{U}_{11}+2\bar{W}_{11}U_{21}), \end{aligned}\tag{3.14a}$$

$$\begin{aligned}\Phi_2 &= \mathrm{i}\alpha_{10}(U_{12}U_{24}+U_{26}U_{11}+U_{23}\bar{U}_{12})+V_{12}U'_{24}+V_{23}\bar{U}'_{12}+V_{24}U'_{12}\\ &\quad+\bar{V}_{12}U'_{23}+V_{11}U'_{26}+\mathrm{i}(\beta_{10}+\beta_{20})\bar{W}_{12}U_{23}+\mathrm{i}(\beta_{10}-\beta_{20})W_{12}U_{24}\\ &\quad+\mathrm{i}\beta_{20}W_{24}U_{12}-\mathrm{i}\beta_{20}W_{23}\bar{U}_{12}+\mathrm{i}\beta_{10}W_{26}U_{11}, \end{aligned}\tag{3.14b}$$

$$\begin{aligned}\Phi_3 &= \mathrm{i}\alpha_{10}(U_{25}W_{11}+2\bar{U}_{11}W_{21}-U_{21}\bar{W}_{11})+V_{11}W'_{25}+V_{21}\bar{W}'_{11}\\ &\quad+\bar{V}_{11}W'_{21}+\mathrm{i}\beta_{10}(W_{25}W_{11}+\bar{W}_{11}W_{21}), \end{aligned}\tag{3.14c}$$

$$\begin{aligned}\Phi_4 &= \mathrm{i}(\alpha_{10}+\alpha_{20})\bar{U}_{12}W_{23}+\mathrm{i}(\alpha_{10}-\alpha_{20})U_{12}W_{24}-\mathrm{i}\alpha_{20}U_{23}\bar{W}_{12}\\ &\quad+\mathrm{i}\alpha_{20}U_{24}W_{12}+\mathrm{i}\alpha_{10}U_{26}W_{11}+V_{12}W'_{24}+V_{23}W'_{12}+V_{24}W'_{12}\\ &\quad+\bar{V}_{12}W'_{23}+V_{11}W'_{26}+\mathrm{i}\beta_{10}(W_{12}W_{24}+W_{26}W_{11}+W_{23}\bar{W}_{12}), \end{aligned}\tag{3.14d}$$

with $\alpha_{10} = 0$. The corresponding expressions for Φ_5, Φ_6, Φ_7, and Φ_8 are obtained from (3.14) with $(\alpha_{10}, \beta_{10}, \Omega_{10})$ and $(\alpha_{20}, \beta_{20}, \Omega_{20})$ interchanged and then α_{10} set equal to zero together with

$$\{A, B, \text{ suffixes } 11, 12, 21, 22, 23, 24, 25, 26\},$$

replaced by

$$\{B, A, \text{ suffixes } 12, 11, 22, 21, 23, 24, 26, 25\}.$$

Finally the terms with suffix 24 are replaced by their complex conjugate. The disturbance velocities (U_{41}, V_{41}, W_{41}) and (U_{42}, V_{42}, W_{42}) must of course vanish at $\zeta = 0$ and the functions $\boldsymbol{S}_{41}, \boldsymbol{S}_{42}$ must match with the corresponding functions in the main deck. The latter matching conditions completely specify inhomogeneous differential equations for $\boldsymbol{S}_{41}$ and $\boldsymbol{S}_{42}$. Since the homogeneous form of these systems have a solution it follows that we must apply solvability conditions to the systems for $\boldsymbol{S}_{41}$, $\boldsymbol{S}_{42}$. In order to write down these conditions we must introduce the differential systems adjoint to those which determine the fundamentals.

Before we describe the adjoint system it is necessary to specify the boundary conditions on the systems for $\boldsymbol{S}_{41}$ and $\boldsymbol{S}_{42}$ as $\zeta \to \infty$. First we must obtain the leading-order solution in the upper deck which in turn will enable the zeroth-order solution in the main deck to be determined. This will produce the required boundary conditions. We are able to consider a single three-dimensional perturbation since the effects from the nonlinear terms in the equations of motion are only present in the lower deck to the order of interest. This formulation will allow the boundary conditions for $\boldsymbol{S}_{41}$ and $\boldsymbol{S}_{42}$ to be obtained by setting $\alpha_0 = 0$ and $\beta = \beta_{10}$ for $\boldsymbol{S}_{41}$ and $\alpha_0 = \alpha_{20}$, $\beta = \beta_{20}$ and $\tilde{\beta} = 0$ for $\boldsymbol{S}_{42}$.

In the upper deck of the triple-deck structure we write the perturbed flow for a three-dimensional disturbance as

$$\boldsymbol{U} = (X, 0, 1) + \epsilon^2(\hat{U}, \hat{V}, \hat{W}), \tag{3.15a}$$

$$P = -\tfrac{1}{2}X^2 + \epsilon^2\hat{P}, \tag{3.15b}$$

where

$$\begin{aligned}(\hat{U}, \hat{V}, \hat{W}, \hat{P}) &= \epsilon^{\frac{3}{4}}E(\hat{U}_1, \hat{V}_1, \hat{W}_1, \hat{P}_1) + \epsilon^{\frac{3}{2}}E^2(\hat{U}_2, \hat{V}_2, \hat{W}_2, \hat{P}_2)\\ &\quad+\epsilon^{\frac{3}{2}}(\hat{U}_M, \hat{V}_M, \hat{W}_M, \hat{P}_M) + \epsilon^{\frac{9}{4}}E(\hat{U}_4, \hat{V}_4, \hat{W}_4, \hat{P}_4) + \ldots + \text{C.C.}, \end{aligned}\tag{3.16}$$

where E is defined by (2.8) and $\hat{P}_1$ is given by (2.13a). We substitute for $\boldsymbol{U}$ and P in the governing equations (2.4) to obtain the solutions for $\hat{U}_4, \hat{V}_4, \hat{W}_4, \hat{P}_4$. We are only interested in the behaviour of $\hat{P}_4$ and $\hat{V}_4$ in the upper deck as $\hat{Y} \to 0$ so we will only give these expressions. We find that at $\hat{Y} = 0$

$$\hat{P}_4 = \frac{a_4(X_n)(\alpha_0 X_n + \beta)^2}{(\alpha_0^2+\beta^2)^{\frac{1}{2}}}, \tag{3.17a}$$

and

$$\hat{V}_4 = -\mathrm{i}(\alpha_0 X_n + \beta)\, a_4 - \frac{\alpha_0(\alpha_0 X_n + \beta)}{(\alpha_0^2+\beta^2)} \frac{\mathrm{d}a}{\mathrm{d}\tilde{X}} + X_n \frac{\mathrm{d}a}{\mathrm{d}\tilde{X}} + \mathrm{i}(\alpha_0 \tilde{X} + \tilde{\beta})\, a - \frac{\mathrm{i}\beta\tilde{\beta}(\alpha_0 X_n + \beta)}{(\alpha_0^2+\beta^2)} a, \tag{3.17b}$$

where $a_4(X)$ is an unknown amplitude function and where some terms arising from the nonlinear interaction between the fundamental and first harmonic terms have been omitted since it is anticipated that they will not be included in the required matching condition.

In the main deck the velocity and pressure fields are expressed in the following way:

$$\boldsymbol{U} = (X\bar{u}, \epsilon^4 \bar{v}, \bar{w}) + (\epsilon\tilde{U}, \epsilon^2 \tilde{V}, \epsilon\tilde{W}), \tag{3.18a}$$

$$P = -\tfrac{1}{2}X^2 + \epsilon^2 \tilde{P}, \tag{3.18b}$$

where $(\tilde{U}, \tilde{V}, \tilde{W}, \tilde{P})$ expand in a similar way to $(\hat{U}, \hat{V}, \hat{W}, \hat{P})$ from (3.16). We find that since $\partial\tilde{P}_4/\partial\eta = 0$ and in order to match with the upper-deck solution as $\eta \to \infty$ the pressure term $\tilde{P}_4$ is given by

$$\tilde{P}_4 = a_4 \frac{(\alpha_0 X_n + \beta)^2}{(\alpha_0^2+\beta^2)^{\frac{1}{2}}}, \tag{3.19a}$$

and from matching the solution for $\tilde{V}_4$ with (3.17b) we have

$$\begin{aligned}\tilde{V}_4 = &-\mathrm{i}a_4(\alpha_0 X_n \bar{u} + \beta\bar{w}) - X_n \bar{u}\frac{\mathrm{d}a}{\mathrm{d}\tilde{X}} - \mathrm{i}(\alpha_0 \tilde{X}\bar{u} + \tilde{\beta}\bar{w})\, a \\ &+ \frac{2X_n(\alpha_0 X_n \bar{u} + \beta\bar{w})}{(\alpha_0 X_n + \beta)} \frac{\mathrm{d}a}{\mathrm{d}\tilde{X}} + \frac{2\mathrm{i}(\alpha_0 X_n \bar{u} + \beta\bar{w})(\alpha_0\tilde{X} + \tilde{\beta})}{(\alpha_0 X_n + \beta)} a \\ &- \frac{\alpha_0(\alpha_0 X_n \bar{u} + \beta\bar{w})}{(\alpha_0^2+\beta^2)} \frac{\mathrm{d}a}{\mathrm{d}\tilde{X}} - \frac{\mathrm{i}\beta(\alpha_0 X_n \bar{u} + \beta\bar{w})}{(a_0^2+\beta^2)} \tilde{\beta}a.\end{aligned} \tag{3.19b}$$

From the continuity equation we can now obtain the behaviour of $\alpha_0 \tilde{U}_4 + \beta\tilde{W}_4$ as $\eta \to 0$, which is the function required for the matching condition between the lower and main decks. Thus, at $\eta = 0$

$$\begin{aligned}\alpha_0 \tilde{U}_4 + \beta\tilde{W}_4 = &\left[\alpha_0 \lambda\tilde{X} + \tilde{\beta}\mu - \frac{2(\alpha_0 X_n \lambda + \beta\mu)(\alpha_0\tilde{X} + \tilde{\beta})}{(\alpha_0 X_n + \beta)} + \frac{(\alpha_0 X_n \lambda + \beta\mu)\beta\tilde{\beta}}{(\alpha_0^2+\beta^2)} - \mu\tilde{\beta}\right] a \\ &+ \left[\frac{2\mathrm{i}X_n(\alpha_0 X_n \lambda + \beta\mu)}{(\alpha_0 X_n + \beta)} - \frac{\mathrm{i}\alpha_0(\alpha_0 X_n \lambda + \beta\mu)}{(\alpha_0^2+\beta^2)}\right] \frac{\mathrm{d}a}{\mathrm{d}\tilde{X}} \\ &+ (\alpha_0 X_n \lambda + \beta\mu)\, a_4.\end{aligned} \tag{3.20}$$

In the main deck the velocity and pressure fields are expressed in a similar way to

(3.15) and (3.16) following (2.10) with $\delta = \epsilon^{\frac{3}{4}}$. Multiplying the X momentum equation by α_0 and the Z momentum equation by β we obtain

$$P_{40} = -\frac{\mathrm{i}}{(\alpha_0^2+\beta^2)}\left(\alpha_0\frac{\partial^2 U_{40}}{\partial\zeta^2}+\frac{\beta\,\partial^2 W_{40}}{\partial\zeta^2}\right)_{\zeta=0} + \frac{1}{(\alpha_0^2+\beta^2)^2}\left(\alpha_0\frac{\mathrm{d}a_l}{\mathrm{d}\tilde{X}}+\mathrm{i}\beta\tilde{\beta}a_l\right)\left(\alpha_0\frac{\partial^2 u_0}{\partial\zeta^2}+\beta\frac{\partial^2 w_0}{\partial\zeta^2}\right)_{\zeta=0}, \tag{3.21}$$

and from matching with the main deck since $\partial P_{40}/\partial\zeta = 0$

$$P_{40} = a_4\frac{(\alpha_0 X_{\mathrm{n}}+\beta)^2}{(\alpha_0^2+\beta^2)^{\frac{1}{2}}}, \tag{3.22}$$

where a_l is the amplitude of the lower-deck disturbance. The matching condition requires that

$$(\alpha_0\tilde{U}_4+\beta\tilde{W}_4)_{\eta=0} = (\alpha_0 U_{40}+\beta W_{40})_{\zeta=\infty}, \tag{3.23}$$

which can be obtained from (3.20), (3.21) and (3.22). After some manipulation we obtain the condition

$$D(\infty)-CD''(0) = (\alpha_0 U_{1l}+\beta W_{1l})_{\zeta=\infty}\left[\frac{2\mathrm{i}X_{\mathrm{n}}}{(\alpha_0 X_{\mathrm{n}}+\beta)}\frac{\mathrm{d}a_l}{\mathrm{d}\tilde{X}} - \frac{2(\alpha_0\tilde{X}+\tilde{\beta})}{(\alpha_0 X_{\mathrm{n}}+\beta)}a_l+\frac{\alpha_0\tilde{X}\lambda}{(\alpha_0 X_{\mathrm{n}}\lambda+\beta\mu)}a_l\right], \tag{3.24a}$$

where

$$D(\zeta) = \alpha_0 U_{40}+\beta W_{40}, \tag{3.24b}$$

$$C = -\frac{\mathrm{i}(\alpha_0 X_{\mathrm{n}}\lambda+\beta\mu)}{(\alpha_0^2+\beta^2)^{\frac{1}{2}}(\alpha_0 X_{\mathrm{n}}+\beta)^2}, \tag{3.24c}$$

and U_{1l} and W_{1l} are defined by $a_l U_{1l} = u_0$ and $a_l W_{1l} = w_0$. We note here that the linear eigenrelation in §2 is $D(\infty)-CF''(0) = 0$ with D given by $D = \alpha_0 u_0+\beta w_0$. We also note that (3.24) can be obtained directly from the linear eigenrelation (2.22) in the form

$$\mathrm{i}(\alpha_0^2+\beta^2)(\alpha_0 X+\beta)^2(\alpha_0 U+\beta W)_\infty = (\alpha_0 X\lambda+\beta\mu)(\alpha_0 U+\beta W)''_0,$$

by writing $X = X_{\mathrm{n}}+\epsilon^{\frac{3}{2}}\tilde{X}$, $\alpha_0 = \alpha_0-\epsilon^{\frac{3}{2}}\mathrm{i}\,\partial/\partial\tilde{X}$, $\beta = \beta+\epsilon^{\frac{3}{2}}\tilde{\beta}$, $U = u_0+\epsilon^{\frac{3}{2}}U_{40}$ and $W = w_0+\epsilon^{\frac{3}{2}}W_{40}$ and equating coefficients of $\epsilon^{\frac{3}{2}}$.

Now we can obtain the solvability conditions to be applied to the systems for $\boldsymbol{S}_{41}$ and $\boldsymbol{S}_{42}$. We first note that if we define

$$F = \alpha_0 u_0+\beta w_0,$$

in (2.18) then the eigenvalue problem which leads to (2.22) can be written as

$$F'''-\mathrm{i}[\alpha_0\lambda X_{\mathrm{n}}+\beta\mu]\zeta F'+\mathrm{i}\Omega F' = 0, \tag{3.25a}$$

$$F(0) = F'(\infty) = 0, \tag{3.25b}$$

$$F(\infty) = CF''(0), \tag{3.25c}$$

where C is given by (3.24c). The system adjoint to (3.25) is

$$p' = 0, \tag{3.26a}$$

$$r''-\mathrm{i}[\alpha_0\lambda X_{\mathrm{n}}+\beta\mu]\zeta r+\mathrm{i}\Omega r = p, \tag{3.26b}$$

$$q(0) = r(\infty) = 0,\quad r(0) = Cp(\infty). \tag{3.26c}$$

It is easily seen that we can take $p = 1$ in (3.26) and then solving the equation for r using variation of parameters and the boundary conditions gives (2.22) again. It then follows that if $\Omega = \Omega(\alpha_0, \beta)$ is an eigenvalue of (3.25) then the system

$$G''' - \mathrm{i}(\alpha_0 X_n \lambda + \mu\beta)\,\zeta G' + \mathrm{i}\Omega G' = R,$$
$$G(0) = G'(\infty) = 0, \quad G(\infty) - CG''(0) = \gamma,$$

will have a solution if
$$\int_0^\infty rR\,\mathrm{d}\zeta = \gamma. \tag{3.27}$$

We will consider in detail the solvability condition applied to the differential system (3.13a). Then it will be a straightforward matter to obtain the corresponding result for (3.13b). If we rearrange (3.13a) in the manner described in §2 in order to obtain the differential equation for $\alpha_0 u_0 + \beta w_0$ then (3.13a) can be expressed as

$$G''' - \mathrm{i}(\alpha_{10}\lambda X_n + \beta_{10}\mu)\,\zeta G' + \mathrm{i}\Omega_{10} G' = Q_1, \tag{3.28a}$$

with boundary conditions
$$G(0) = G'(\infty) = 0, \tag{3.28b}$$

and
$$G(\infty) - CG''(0) = \gamma_1, \tag{3.28c}$$

where now $G = \alpha_{10} U_{41} + \beta_{10} W_{41}$. The function Q_1 is given by

$$Q_1 = (\alpha_{10}\lambda X_n + \beta_{10}\mu) R_2 + \alpha_{10}\frac{\mathrm{d}R_1}{\mathrm{d}\zeta} + \beta_{10}\frac{\mathrm{d}R_3}{\mathrm{d}\zeta}, \tag{3.28d}$$

where R_1, R_2 and R_3 are the right-hand sides of (3.13a). From (3.24a) we have

$$\gamma_1 = (\alpha_{10} U_{11} + \beta_{10} W_{11})_\infty \left[\frac{2\mathrm{i}X_n}{(\alpha_{10} X_n + \beta_{10})}\frac{\mathrm{d}A}{\mathrm{d}\tilde{X}} - \frac{2(\alpha_{10}\tilde{X} + \tilde{\beta})}{(\alpha_{10} X_n + \beta_{10})}A + \frac{\alpha_{10}\tilde{X}\lambda}{(\alpha_{10} X_n \lambda + \beta_{10}\mu)}A\right]. \tag{3.28e}$$

Note that $\alpha_{10} = 0$ in the above equations but has been included so that the corresponding solutions for $\boldsymbol{S}_{42}$ may be readily obtained. After some rearrangement, applying the solvability condition (3.27) to the above system (3.28) and the corresponding one for $\boldsymbol{S}_{42}$, produces a pair of amplitude equations for A and B.

Thus it follows that the differential systems for $\boldsymbol{S}_{41}, \boldsymbol{S}_{42}$ will have a solution if

$$\frac{\mathrm{d}A}{\mathrm{d}\tilde{X}} = \lambda_1 \tilde{\beta} A - a_1 A|A|^2 - b_1 A|B|^2, \tag{3.29a}$$

$$\frac{\mathrm{d}B}{\mathrm{d}\tilde{X}} = \lambda_2 \tilde{X} B - a_2 B|B|^2 - b_2 B|A|^2. \tag{3.29b}$$

Here the coefficients λ_1, λ_2 are defined by

$$\lambda_1 = \frac{-2\mathrm{i}}{3\beta_{10}}[2 + H_1] \Big/ \left[X_n G_1 + \frac{2X_n \lambda H_1}{3\beta_{10}\mu}\right], \tag{3.30a}$$

$$\lambda_2 = -\left[\alpha_{20} G_2 + \frac{2\alpha_{20}\lambda H_2}{3(\alpha_{20} X_n \lambda + \beta_{20}\mu)}\right] \Big/ \left[X_n G_2 + \frac{\alpha_{20}}{\alpha_{20}^2 + \beta_{20}^2} + \frac{2X_n \lambda H_2}{3(\alpha_{20} X_n \lambda + \beta_{20}\mu)}\right], \tag{3.30b}$$

where for $n = 1, 2, G_n, H_n$ are defined by

$$G_n = \frac{2}{\alpha_{n0} X_n + \beta_{n0}} - \frac{5\lambda}{3(\alpha_{n0} X_n \lambda + \beta_{n0} \mu)},$$

$$H_n = \xi_{0n} \operatorname{Ai}(\xi_{0n}) \left(\frac{\xi_{0n}}{\operatorname{Ai}'(\xi_{0n})} + \frac{1}{\chi_0(\xi_{0n})} \right),$$

with

$$\xi_{0n} = \frac{-\mathrm{i}^{\frac{1}{3}} \Omega_{n0}}{(\alpha_{n0} X_n \lambda + \beta_{n0} \mu)^{\frac{2}{3}}}, \quad \alpha_{10} = 0.$$

Finally the constants a_1, b_1, a_2, b_2 are defined by

$$a_1 = \beta_{10} \int_0^\infty r_1 \Phi_3' \, \mathrm{d}\zeta \bigg/ \bigg[\beta_{10} \lambda X_n \int_0^\infty \zeta r_1 W_{11}' \, \mathrm{d}\zeta + \beta_{10} \int_0^\infty r_1 (\lambda X_n W_{11} - \mu U_{11}) \, \mathrm{d}\zeta - 2\mathrm{i} W_{11}(\infty) X_n \bigg],$$

$$b_1 = \beta_{10} \int_0^\infty r_1 \Phi_4' \, \mathrm{d}\zeta \bigg/ \bigg[\beta_{10} \lambda X_n \int_0^\infty \zeta r_1 W_{11}' \, \mathrm{d}\zeta + \beta_{10} \int_0^\infty r_1 (\lambda X_n W_{11} - \mu U_{11}) \, \mathrm{d}\zeta - 2\mathrm{i} W_{11}(\infty) X_n \bigg],$$

$$a_2 = \int_0^\infty r_2 [\alpha_{20} \Phi_5' + \beta_{20} \Phi_7'] \, \mathrm{d}\zeta \bigg/ \bigg[\lambda X_n \int_0^\infty \zeta r_2 [\alpha_{20} U_{12}' + \beta_{20} W_{12}'] \, \mathrm{d}\zeta + \beta_{20} \int_0^\infty r_2 (\lambda X_n W_{12} - \mu U_{12}) \, \mathrm{d}\zeta - 2\mathrm{i} [\alpha_{20} U_{12} + \beta_{20} W_{12}]_\infty \frac{X_n}{\alpha_{20} X_n + \beta_{20}} \bigg],$$

$$b_2 = \int_0^\infty r_2 [\alpha_{20} \Phi_6' + \beta_{20} \Phi_8'] \, \mathrm{d}\zeta \bigg/ \bigg[\lambda X_n \int_0^\infty \zeta r_2 [\alpha_{20} U_{12}' + \beta_{20} W_{12}'] \, \mathrm{d}\zeta + \beta_{20} \int_0^\infty r_2 (\lambda X_n W_{12} - \mu U_{12}) \, \mathrm{d}\zeta - 2\mathrm{i} [\alpha_{20} U_{12} + \beta_{20} W_{12}]_\infty \frac{X_n}{\alpha_{20} X_n + \beta_{20}} \bigg],$$

where r_1 and r_2 correspond to r in (3.26) with $(\alpha, \beta, \Omega) = (0, \beta_{10}, \Omega_{10})$ and $(\alpha, \beta, \Omega) = (\alpha_{20}, \beta_{20}, \Omega_{20})$ respectively.

4. The generalization and solution of the amplitude equations

First we note that the three-dimensional wave can also be 'de-tuned' by varying β_1 by an amount $\epsilon^{\frac{3}{2}} \tilde{\tilde{\beta}}$ from the neutral value. In that case (3.29) becomes

$$\frac{\mathrm{d}A}{\mathrm{d}\tilde{\tilde{X}}} = \lambda_1 \tilde{\beta} A - a_1 A |A|^2 - b_1 A |B|^2, \tag{4.1a}$$

$$\frac{\mathrm{d}B}{\mathrm{d}\tilde{\tilde{X}}} = \lambda_3 \tilde{\tilde{\beta}} B + \lambda_2 \tilde{X} B - a_2 B |B|^2 - b_2 B |A|^2, \tag{4.1b}$$

where λ_3 is defined by an expression similar to (3.30a). Secondly we note that (3.29) apply in a $\epsilon^{\frac{3}{2}}$ neighbourhood of the position where the three-dimensional wave is neutrally stable. Following Hall & Smith (1984) it can be shown that (3.29) apply over a longer lengthscale if $\lambda_2 \tilde{X}$ in (4.1*b*) is replaced by $\lambda_2 \tilde{\tilde{X}}$, where $\tilde{\tilde{X}}$ is then treated as a constant in the amplitude equations. This result can be found directly from (4.1) by letting $\tilde{X} \rightarrow \infty$ and introducing a lengthscale shorter than $\tilde{X}$ in order to retain the derivative terms. The resulting amplitude equations have a 'quasi-parallel' nature and correspond to the calculation of Smith (1979*b*).

We now define ρ and σ by

$$\rho = |A|^2, \quad \sigma = |B|^2,$$

in which case (3.29) and the generalization of this system for $\tilde{X} \gg 1$ can be written

$$\rho_{\tilde{X}} = 2\rho\{\lambda_{1r}\tilde{\beta} - a_{1r}\rho - b_{1r}\sigma\}, \tag{4.2a}$$

$$\sigma_{\tilde{X}} = 2\sigma\{\lambda_{2r}\tilde{X} - a_{2r}\sigma - b_{2r}\rho\}, \tag{4.2b}$$

and

$$\rho_{\tilde{X}} = 2\rho\{\lambda_{1r}\tilde{\beta} - a_{1r}\rho - b_{1r}\sigma\}, \tag{4.3a}$$

$$\sigma_{\tilde{X}} = 2\sigma\{\lambda_{3r}\tilde{\tilde{\beta}} + \lambda_{2r}\tilde{\tilde{X}} - a_{2r}\sigma - b_{2r}\rho\}. \tag{4.3b}$$

The precise nature of the solutions of (4.2), (4.3) depends sensitively on the constants appearing in these equations. We shall see in the next section that the constants a_{1r} and a_{2r} are positive almost everywhere so we first discuss such a situation in detail. In fact a_{1r} is always positive and this result is entirely consistent with the finite-Reynolds-number calculations of HM for the two-dimensional mode.

A matter of some importance is the question of whether ρ or σ in (4.2) or (4.3) can become infinite at a finite value of $\tilde{X}$. This would mean that three-dimensionality could destroy the stable equilibrium states of HM. We seek a singularity of either system as $\tilde{X} \rightarrow \tilde{X}_0$ by writing

$$\rho = \frac{\rho_0}{(\tilde{X}_0 - \tilde{X})} + \ldots, \quad \sigma = \frac{\sigma_0}{(\tilde{X}_0 - \tilde{X})} + \ldots,$$

in which case ρ_0, σ_0 satisfy

$$1 = 2(-a_{1r}\rho_0 - b_{1r}\sigma_0), \tag{4.4a}$$

$$1 = 2(-a_{2r}\sigma_0 - b_{2r}\rho_0), \tag{4.4b}$$

and ρ_0 and σ_0 must of course both be positive. It follows immediately that no such singularity is possible if a_{1r}, a_{2r}, b_{1r} and b_{2r} are all positive. In fact it is easily shown that with a_{1r} and a_{2r} positive the only case when the singularity can occur is when b_{1r} and b_{2r} are negative and

$$a_{1r}a_{2r} < b_{1r}b_{2r}. \tag{4.5}$$

This condition effectively identifies an important class of three-dimensional waves which can have a significant effect on the two-dimensional equilibrium states of HM. In order to see why this is the case it is necessary for us to discuss the solutions of (4.2) and (4.3) in more detail. We continue to discuss the solution for the case when a_{1r} and a_{2r} are both positive.

In fact we begin with a discussion of (4.3) and return to (4.2) later. It is easily shown from (4.3) that ρ and σ have the possible equilibrium states:

(*a*) $\rho = \sigma = 0,$ (4.6*a*)

(*b*) $\rho = \lambda_{1r}\tilde{\beta}a_{1r}^{-1}, \quad \sigma = 0,$ (4.6*b*)

(*c*) $\rho = 0, \quad \sigma = [\lambda_{3r}\tilde{\tilde{\beta}} + \lambda_{2r}\tilde{\tilde{X}}]a_{2r}^{-1},$ (4.6*c*)

(*d*) $\rho = [\lambda_{1r}\tilde{\beta} - b_{1r}\sigma]a_{1r}^{-1}, \quad \sigma = \{\lambda_{1r}\tilde{\beta}b_{2r} - [\lambda_{3r}\tilde{\tilde{\beta}} + \lambda_{2r}\tilde{\tilde{X}}]a_{1r}\}/\{b_{1r}b_{2r} - a_{1r}a_{2r}\}.$ (4.6*d*)

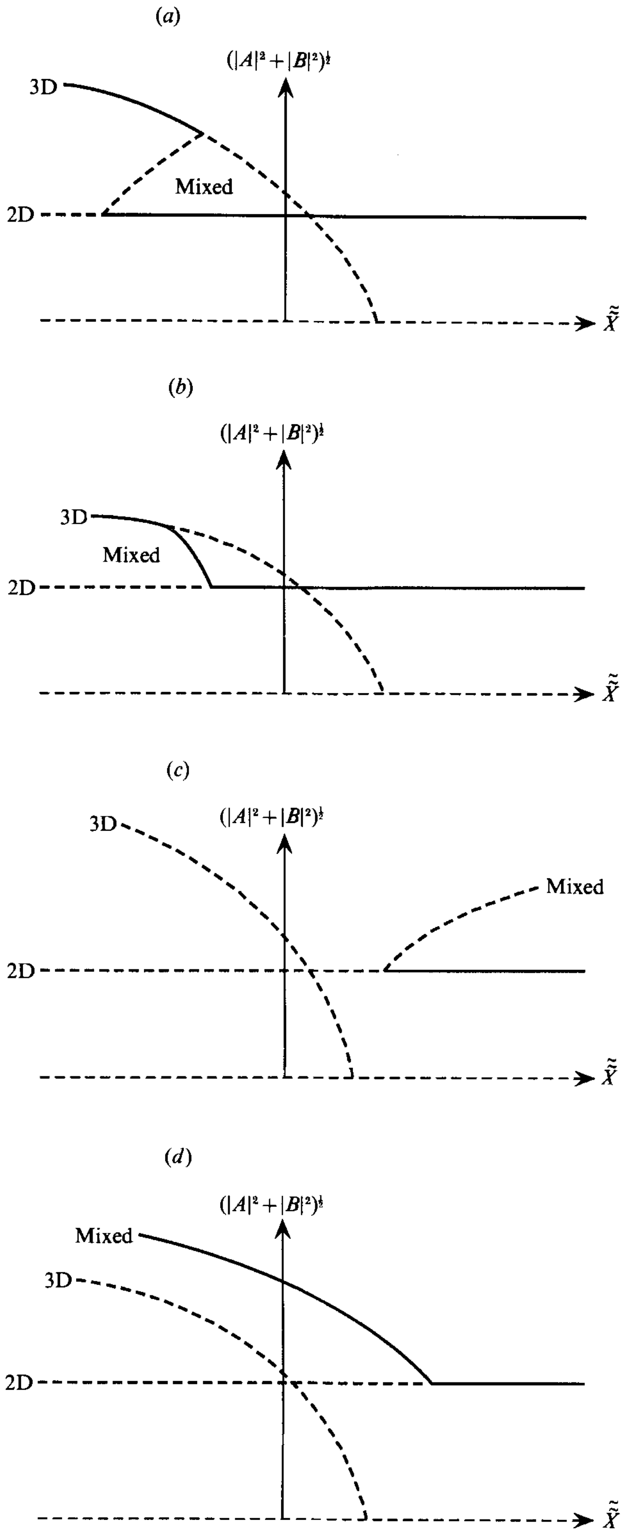

FIGURE 3(a–d). For caption see facing page.

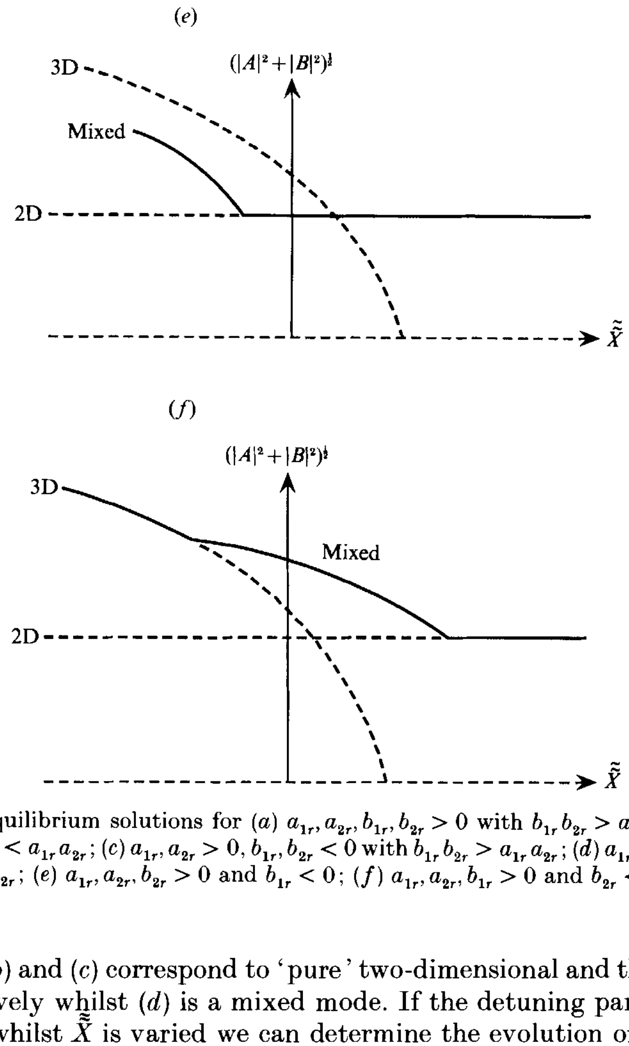

FIGURE 3. The equilibrium solutions for (a) $a_{1r}, a_{2r}, b_{1r}, b_{2r} > 0$ with $b_{1r} b_{2r} > a_{1r} a_{2r}$; (b) $a_{1r}, a_{2r}, b_{1r}, b_{2r} > 0$ with $b_{1r} b_{2r} < a_{1r} a_{2r}$; (c) $a_{1r}, a_{2r} > 0$, $b_{1r}, b_{2r} < 0$ with $b_{1r} b_{2r} > a_{1r} a_{2r}$; (d) $a_{1r}, a_{2r} > 0$, $b_{1r}, b_{2r} < 0$ with $b_{1r} b_{2r} < a_{1r} a_{2r}$; (e) $a_{1r}, a_{2r}, b_{2r} > 0$ and $b_{1r} < 0$; (f) $a_{1r}, a_{2r}, b_{1r} > 0$ and $b_{2r} < 0$.

The solutions (b) and (c) correspond to 'pure' two-dimensional and three-dimensional modes respectively whilst (d) is a mixed mode. If the detuning parameters $\tilde{\beta}$ and $\tilde{\tilde{\beta}}$ are held fixed whilst $\tilde{\tilde{X}}$ is varied we can determine the evolution of the equilibrium amplitudes as the disturbance develops away from the attachment line $X_{\mathrm{n}} = 0$. For each case the phase-plane solutions were plotted for ρ and σ. The stability of the two-dimensional mode, the three-dimensional mode and the mixed mode, corresponding to (4.6b–d) are determined as $\tilde{\tilde{X}}$ is increased from $-\infty$. Note that from the calculations λ_{2r} from (3.30b) is negative. The stability of the different equilibrium solutions can be checked by a routine stability analysis. Before discussing the nature of the solutions we note that in all the cases we computed a_{1r} and a_{2r} are almost always positive so that nonlinear effects are stabilizing if either the two- or three-dimensional modes exists separately. We further assume that the detuning parameter $\tilde{\beta}$ has been chosen such that $\lambda_{1r}\tilde{\beta}a_{1r} > 0$ so that in the absence of a three-dimensional wave a stable finite-amplitude wave propagating along the attachment line is possible. If a_{1r} and a_{2r} are positive then there are four possible combinations of signs for b_{1r} and b_{2r}. The bifurcation properties for these four cases are summarized below. In figure 3(a–f) described below the solid lines denote stable solutions while the broken ones denote unstable solutions.

Case I: $a_{1r}, a_{2r}, b_{1r}, b_{2r} > 0$

The different possible solutions in this case are shown in figures 3(*a*) and 3(*b*) for the 'sub-cases' $b_{1r} b_{2r} > a_{1r}, a_{2r}$ and $b_{1r} b_{2r} < a_{1r} a_{2r}$ respectively. Sufficiently far upstream we see that only the pure two-dimensional mode is a possible stable mode whilst sufficiently far downstream only the two-dimensional mode is a possible equilibrium flow. In the case $b_{1r} b_{2r} < a_{1r} a_{2r}$, there is a short interval where the mixed mode is the only possible stable state.

Case II: $a_{1r}, a_{2r} > 0$, $b_{1r}, b_{2r} < 0$

The solutions in this case are shown in figures 3(*c*) and 3(*d*) for the 'sub-cases' $b_{1r} b_{2r} > a_{1r} a_{2r}$ and $b_{1r} b_{2r} < a_{1r} a_{2r}$, respectively. In the first case the only stable solution is the two-dimensional mode beyond the position where the mixed mode bifurcates from it. However, a phase-plane analysis shows that a sufficiently large disturbance to this state is unstable. Thus there is a threshold type of response where a small disturbance to the two-dimensional mode decays whilst a sufficiently large one will grow. The size of the 'sufficiently large disturbance' decreases to zero as $\tilde{\tilde{X}}$ decreases to the point where the mixed mode bifurcates. Before this point there are no stable modes and any disturbance will grow; in this case and the threshold-amplitude case the growing disturbances terminate in the finite-$\tilde{X}$ singularity discussed previously.

Case III: $a_{1r}, a_{2r}, b_{2r} > 0$, $b_{1r} < 0$

Here the situation is as illustrated in figure 3(*e*). Dependent on the value of $\tilde{\tilde{X}}$ either the mixed or two-dimensional mode is stable. A phase-plane analysis shows that each stable state is stable to an arbitrarily large disturbance so there is no threshold-amplitude type of response.

Case IV: $a_{1r}, a_{2r}, b_{1r} > 0$, $b_{2r} < 0$

The situation is now virtually the same as Case III except that mixed mode loses stability to the three-dimensional mode when $\tilde{\tilde{X}}$ decreases, so that the three-dimensional mode is stable as $\tilde{\tilde{X}} \to -\infty$. Again there is no threshold-amplitude type of response at any value of $\tilde{\tilde{X}}$. This is illustrated in figure 3(*f*).

Thus we see that apart from the case $a_{1r}, a_{2r} > 0$, $b_{1r}, b_{2r} < 0$ with $b_{1r} b_{2r} > a_{1r} a_{2r}$ there is always a stable equilibrium state available at any value of $\tilde{\tilde{X}}$. Furthermore, apart from the case just mentioned, at sufficiently negative values of $\tilde{\tilde{X}}$ the stable state is never the two-dimensional mode. However, as the disturbance develops with increasing $\tilde{\tilde{X}}$ ultimately only the two-dimensional mode is stable. In the exceptional case a sufficiently large initial disturbance will terminate in a singularity at a finite value of $\tilde{\tilde{X}}$.

We now turn to the case where a_{1r} and a_{2r} are not both positive. We shall see in the next section that this situation is unusual and occurs when the constant a_{2r} becomes negative so that nonlinear effects destabilize the three-dimensional mode. The situation in this case can be investigated following the previous discussion. The main result is that (4.1) then always permits a solution which becomes infinite at a finite value of $\tilde{X}$. The singularity has the same structure as that discussed above with the only change being that, dependent on the other constants, it is possible for B alone to become infinite. The equilibrium solutions of the amplitude equations and thus instability characteristics can similarly be investigated for the case $a_{2r} < 0$. Here the three-dimensional mode bifurcates to the right and is always unstable. In some situations the mixed mode exists and it is possible for the two-dimensional mode to be stable to small perturbations. However, sufficiently large perturbations always destabilize the flow so that we conclude that when $a_{2r} < 0$ the presence of sufficiently large-amplitude perturbations will always lead to the finite-$\tilde{X}$ singularity being set up. We conclude that there are just two situations where the ultimate state set up after a wave interaction between two- and three-dimensional modes will not be a stable two-dimensional mode. These exceptional circumstances correspond to when $a_{1r}, a_{2r} > 0$, $b_{1r}, b_{2r} < 0$ with $a_{1r} a_{2r} < b_{1r} b_{2r}$ or whenever $a_{2r} < 0$.

A similar type of discussion for (4.2) is not possible because there are no equilibrium states for this system for all $\tilde{X}$. However, for large values of $\tilde{X}$ it is easy to show that there is a solution with $\rho = \lambda_{1r} \tilde{\beta} a_{1r}^{-1}$, $\sigma = 0$ and that this solution is stable. There are no other equilibrium states so that, unless limit-cycle solutions of (4.2) exist, or a singularity develops we expect any initial disturbance to evolve into a pure two-dimensional mode at large $\tilde{X}$. Numerical investigation of (4.2) showed no evidence of limit-cycle behaviour and that in the exceptional case a finite-$\tilde{X}$ singularity develops and the two-dimensional equilibrium state is then never set up. It remains for us to discuss the values of $a_{1r}, a_{2r}, b_{1r}, b_{2r}$ found in our calculations so that the above results can be applied to the instability of attachment-line flow.

5. Results and discussion

We have seen in the previous section that the nature of the solutions of the amplitude equations depends crucially on the constants $a_{1r}, b_{1r}, a_{2r}, b_{2r}$. These constants can be found only after the differential systems for the fundamentals, adjoint, first harmonic, mean flow correction functions have been solved numerically. These systems were solved using finite differences in the manner described in Hall & Smith (1984); the reader is referred to that paper for a more detailed description of the method. It was found to be convenient to map the region $0 < \zeta < \infty$ into $[0, 1]$ using the transformation

$$\eta = \frac{2}{\pi} \tan^{-1} \zeta,$$

which aids the convergence of the velocity field at large ζ. The other significant difference between our calculations and those of Hall & Smith is that here the spanwise momentum equation has a solution with the velocity component tending to a constant rather than decaying algebraically to zero. In order to illustrate how this can be taken into account we consider the equations for $U_{12}, V_{12}, W_{12}, P_{12}$. By combining the X^* and Z momentum equations we can show that $F = \alpha_{20} U_{12} + \beta_{20} W_{12}$ and $G = \mu U_{12} - \lambda X_{\mathrm{n}} W_{12}$ satisfy

$$F''' - [-\mathrm{i}\Omega_{20} + \mathrm{i}(\lambda X_{\mathrm{n}} \alpha_{20} + \mu\beta_{20}) \zeta] F' = 0, \tag{5.1a}$$

$$G'' - [-\mathrm{i}\Omega_{20} + \mathrm{i}(\lambda X_{\mathrm{n}} \alpha_{20} + \mu\beta_{20}) \zeta] G = i(\alpha_{20} \mu - \lambda X_{\mathrm{n}} \beta_{20}) P_{12}. \tag{5.1b}$$

The first of these equations is to be solved such that $F(0) = F'(0)$ and $F \rightarrow$ constant when $\zeta \rightarrow \infty$ whilst the second equation is solved subject to $G(0) = 0, G(\infty) \sim \zeta^{-1}$. Thus the combination $\mu U_{12} - \lambda X_{\mathrm{n}} W_{12}$ decay algebraically when $\zeta \rightarrow \infty$. Once the equations are solved for F and G we can determine U_{12} and W_{12} and then the equation of continuity is solved to determine V_{12}. The equation for the first harmonic functions can be integrated using the same procedure. Finally in our discussion of the

α_{20}	β_{20}	Ω_{20}	$-a_{1r}$	b_{1r}	a_{2r}	b_{2r}
−0.1614	0.0427	0.0615	−15.5854	−32081339.0	−12895.0	−80.4869
−0.3162	0.1142	0.2019	−15.5801	−159114.0	−757.2574	100.4104
−0.3747	0.1933	0.3674	−15.5801	41.1670	−177.9206	300.6889
−0.3725	0.2523	0.4879	−15.5801	1314.9614	−89.1333	202.8252
−0.3443	0.3110	0.6040	−15.5801	666.9944	−52.8080	−45.7879
0.0490	0.5000	1.0095	−15.5801	−74.2458	−14.8748	2.3777
0.3011	0.4668	1.0375	−15.5801	10.6631	−13.9080	−73.8144
0.5251	0.3795	0.9864	−15.5801	334.7004	−15.7352	113.9694
0.9849	0.2020	0.8797	−15.5801	368.3667	−19.8390	692.2803
3.4675	−0.1164	1.1672	−15.5801	675.7219	1.1880	−2153.2899
5.0859	−0.2706	1.3970	−15.5801	231.6547	2.0798	−2059.4528
10.6798	−0.7970	2.0859	−15.5801	36.0614	0.7039	−1666.8360

TABLE 1. Typical neutral values for $X_n = 0.1$

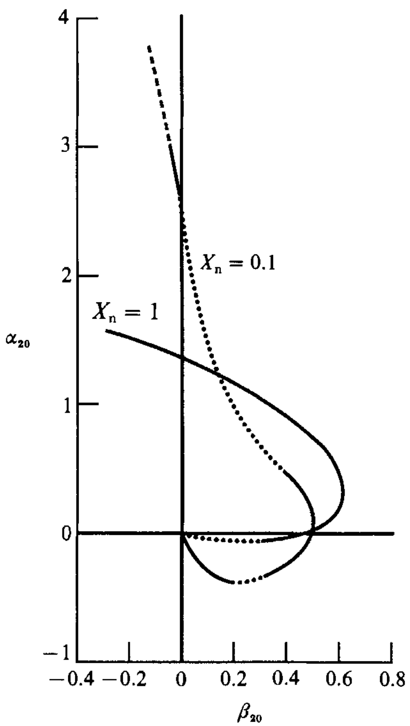

FIGURE 4. The different bifurcation solutions for $X_n = 0.1, 1$.

numerical scheme we note that the convergence of our scheme was checked when appropriate by varying the step length over ζ_∞, the approximation to ∞ in the ζ-direction.

The constants a_1, b_1, a_2 and b_2 were calculated for $X_n = 0.1, 1, 5$ and 10. The results are normalized by making $\alpha_{10}\,U''_{11}+\beta_{10}\,W''_{11}$ and $\alpha_{20}\,U''_{12}+\beta_{20}\,W''_{12}$ both equal to unity at $\zeta = 0$. Some typical values of these constants are shown in table 1 for the exceptional case. We see that it is possible for either of the two exceptional cases of the previous solution to occur. In figures 4 and 5 we have plotted the neutral values of α_{20}, β_{20} and indicated where the exceptional cases occur. The first exceptional case

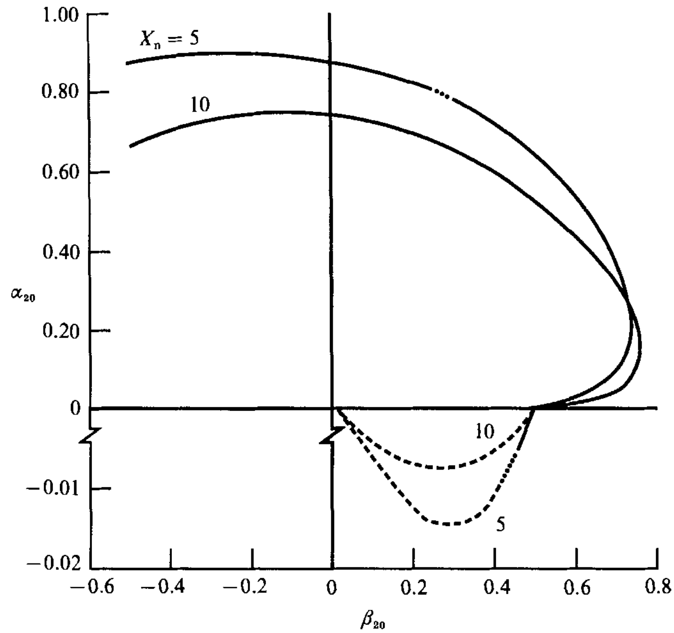

FIGURE 5. The different bifurcation solutions for $X_n = 5$, 10.

with $a_{1r}, a_{2r} > 0$ is denoted by the dotted line whilst the other exceptional case is denoted by the dashed line.

We see that at $X_n = 0.1$ an interaction of the two-dimensional mode with the three-dimensional mode with $\alpha/\beta > \sim 1.27$ will cause a singularity in the disturbance amplitudes to occur. Thus at $X_n = 0.1$ three-dimensional waves propagating at an angle of more than about 50° to the attachment line will cause the catastrophic breakdown of the two-dimensional mode.

A further band of modes with $\alpha_{20} < 0$ which leads to the first exceptional case is also seen to exist. These correspond to low-frequency three-dimensional modes. In the limit as $\alpha_{20} \to 0$ these modes have zero effective shear stress and correspond to the stationary viscous crossflow modes of Hall (1986) and MacKerrell (1987). We conclude that near the attachment line the stimulation of oblique waves propagating at an angle greater than about 50° or the stimulation of the viscous crossflow modes of Hall and MacKerrell will cause a new larger amplitude disturbance flow structure to develop.

When $X_n = 1$ only the destabilizing band of wavenumbers corresponding to the low-frequency modes remains and the interval over which they exist has decreased. However, when $X_n = 5$ the stationary viscous crossflow modes becomes subcritically unstable so that the stationary viscous crossflow modes cause the finite-$\tilde{X}$ singularity to develop at almost all of the possible negative values of α_{20}. In addition there is a very short band of oblique modes propagating at an angle of about 80° to the attachment line which leads to the singularity being set up. This band of unstable wavelengths no longer occurs at $X_n = 10$ but the stationary viscous crossflow modes are now subcritically unstable for almost all of the possible values of α_{20} with $\alpha_{20} < 0$.

Without prohibitively expensive numerical calculations we cannot confirm that the results discussed above show the overall trend of the possible interactions when

X_n increases. In fact some further investigation showed that the small band of destabilizing oblique modes at X_n appears and disappears as X_n varies. However, our calculations do suggest that at small values of X_n there is a wide range of possible oblique modes and a small band of low-frequency modes which, if excited, will cause a catastrophic breakdown of the disturbance flow field. Further away from the attachment line the oblique modes become less important and it is the lower frequency modes which become the dominant mechanism.

Clearly our analysis cannot predict what kind of flow will be set up once the singularity appears. However we note that other modes, notably the inviscid stationary crossflow vortex mode of Gregory *et al.* (1955) might then become important.

This research was supported by the National Aeronautics and Space Administration under NASA Contract No. NAS1-18107 while the authors were in residence at the Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Centre, Hampton, VA 23665. The work was carried out while the second author was in receipt of an SERC research studentship.

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